Chapter 9
1D Fourier Imaging, k-Space and Gradient echoes

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- **Previous sessions:**
  - MR signal detection (Chap.7)
  - T2* (Chap.8)
- **Today's content**
  - Signal and effective spin density
  - Frequency encoding and 1D Fourier transform
  - Gradient echo
  - k-Space

Signal and Effective Spin Density

- $\omega(\vec{r}) = \gamma(B_0 + \Delta B(\vec{r}))$

- Local, microscopic field inhomogeneity T2*

- With an globally well-defined spatial field variation (e.g. linear gradient field), spatial information can also be represented as frequency components

- MR Imaging is to reconstruct the spatial distribution of the sample spin density from such signal encoded with spatial information

Signal and Effective Spin Density

- Effective spin density

$$S(t) = \int d^3 r \rho(\vec{r}) e^{i(\Omega t + \phi(\vec{r},t))} \quad \text{(ref Eq. 7.28)}$$

$$\rho(\vec{r}) \equiv \omega_0 \Lambda \mathcal{B}_\perp M_0(\vec{r}) = \frac{1}{4} \omega_0 \Lambda \mathcal{B}_\perp \rho_0(\vec{r}) \frac{\gamma^2 h^2}{kT} B_0$$

  - Λ represents the total effects from electronic detection system
  - All ‘fields’ are considered spatially and temporally homogeneous, and relaxation effects are ignored
  - Effective spin density is proportional to actual proton density of the sample, while being modulated by many factors
  - For convenient, we’ll simply use ‘spin density’ for $\rho(\vec{r})$

Fourier Transform

- Definition

$$FT[f(x)] = F(k) = \int dx f(x) e^{-i2\pi kx}$$

$$IFT[F(k)] = f(x) = \int dk F(k) e^{i2\pi xk}$$

x and k are Fourier conjugates, representing a pair of FT domains

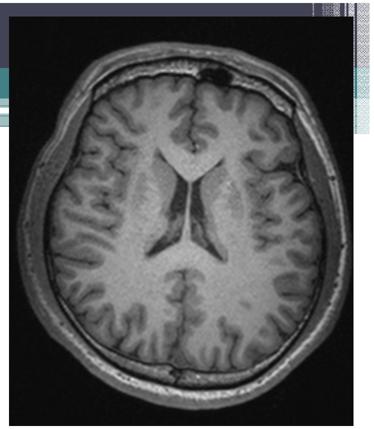
Example:

Time (t) Frequency (f)
Distance (x) Spatial Frequency (1/x)

Property:

f(x) = IFT[FT[f(x)]]

Frequency encoding



- **Goal of MR imaging:**
  Determine the spatial distribution of (effective) spin density of the sample

- **Means:**
  Spatially encodes the (effective) spin density by a linearly varying field, e.g. $G_z(t)$ $B_z/$ z, so that:

$$B_{z(z,t)} = B_0 + zGz(t)$$

$$\Downarrow$$

$$\omega_{z(z,t)} = \gamma B_{z(z,t)} = \omega_0 + \gamma z G_z(t)$$

$$\Downarrow \text{ Rotating frame}$$

$$\phi_{G(z,t)} = -\gamma z \int_0^t dt' G(t')$$

The use of a gradient to establish a relation between the position of spins along some direction and their precessional rates is referred to as frequency encoding along that direction

Frequency encoding

- Signal at the presence of $G_z(t)$

$$s(t) = \int dz\rho(z)e^{i\phi_G(z,t)}$$

$$= \int dz\rho(z)e^{-i2\pi z(\gamma \int_0^t dt' G(t'))}$$

$$\boxed{k(t) \,\square\, \gamma \int_0^t dt' G(t'))}$$

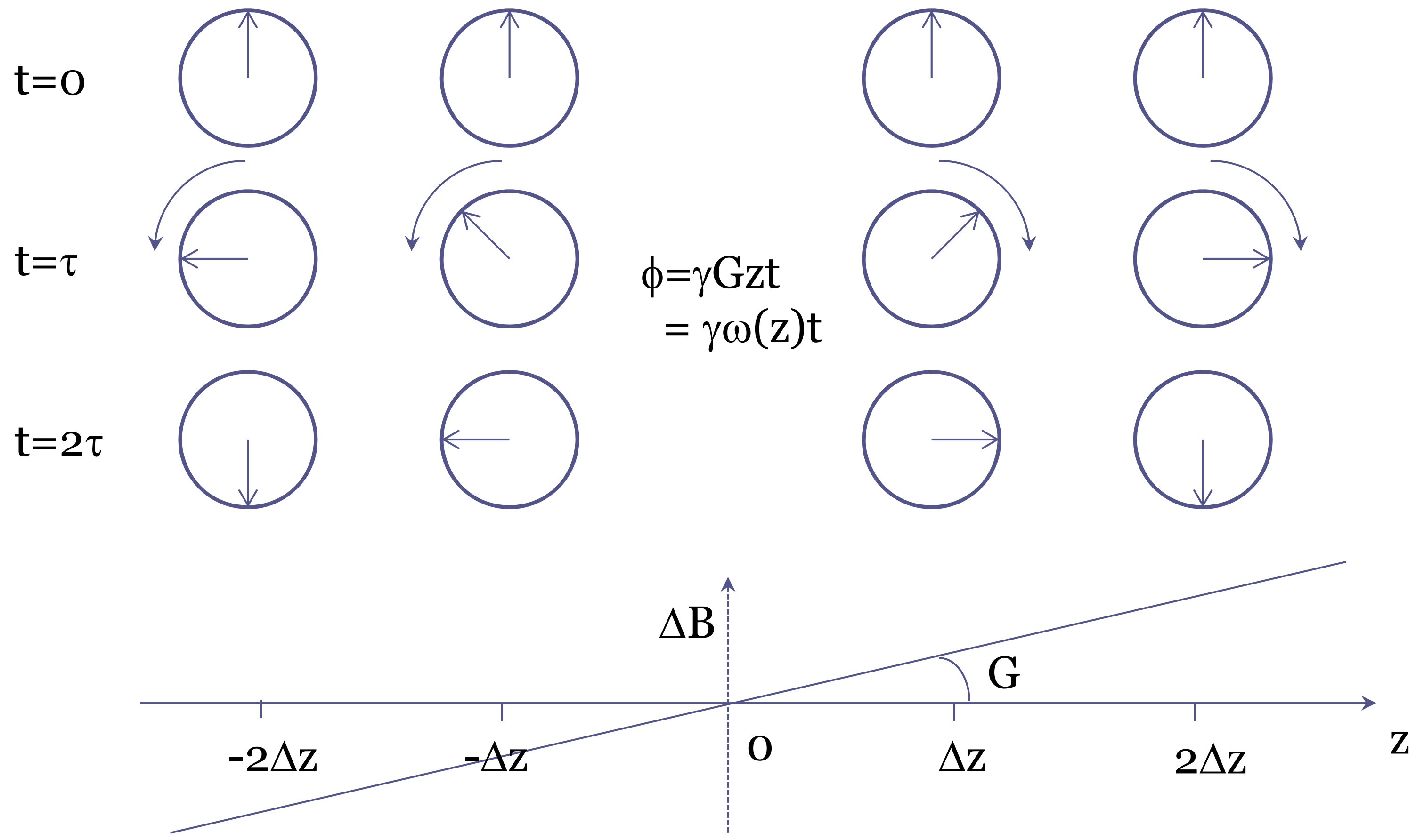
$$s(k) = \int dz\rho(z)e^{-i2\pi zk} \Rightarrow \rho(z) = \int dk s(k)e^{+i2\pi zk}$$

Fourier transform

Invert Fourier transform

Frequency encoding

Consider:
Two spins at z1 and z2, but with precessing freq difference greater than G|z1-z2|



What is k-space?

- A mathematically defined Fourier space for the spatial frequency (how come?)

$$k(t) \square \bar{\gamma} \int_0^t dt' G(t'))$$

- Each S(k) contains signal contribution from all excited spins in the sample, albeit the contribution is weighted by spin density and phase

$$s(k) = \int dz \rho(z) e^{-i2\pi zk}$$

Dirac Delta Function

$$\begin{cases} \delta(z-a) = 0, & z \neq a \\ \int_{a-\varepsilon}^{a+\varepsilon} dz\delta(z-a) = 1, & \varepsilon \to 0 \end{cases}$$

- Properties

$$\int_{-\infty}^{\infty} dz\delta(z-a)f(z) = f(a)$$

$$\delta(z-a) = \int_{-\infty}^{\infty} dk\, e^{i2\pi k(z-a)} = IFT[e^{-i2\pi ka}]$$

$$FT[\delta(z-a)] = e^{-i2\pi ka}$$

Note the periodicity

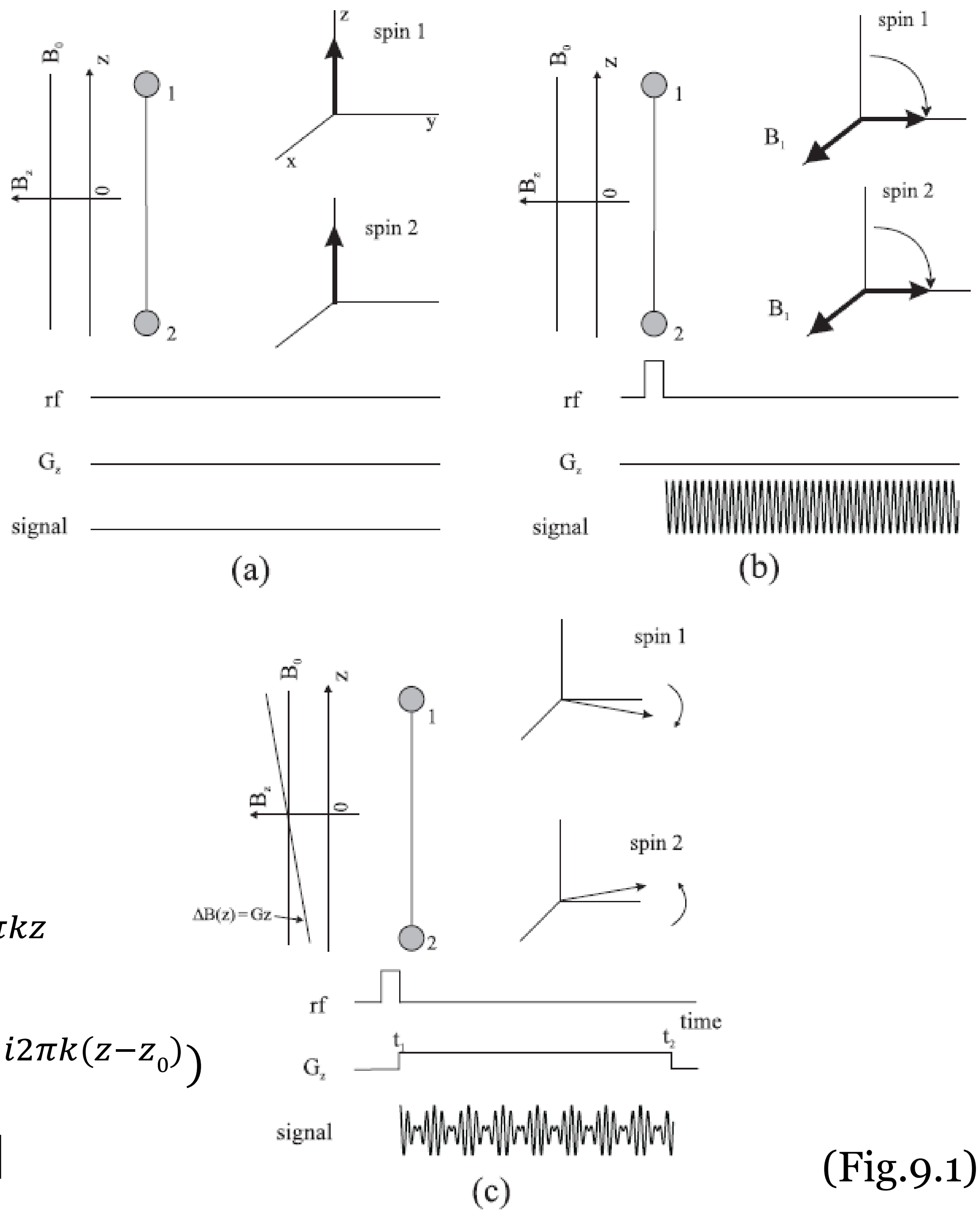
Two spin example analysis

$$s(t) = S_0\left(e^{-i\gamma G z_0 t} + e^{i\gamma G z_0 t}\right)$$
$$= 2S_0\cos(\gamma G z_0 t)$$

$$s(k) = 2S_0\cos(2\pi k z_0)$$

$$\Downarrow$$

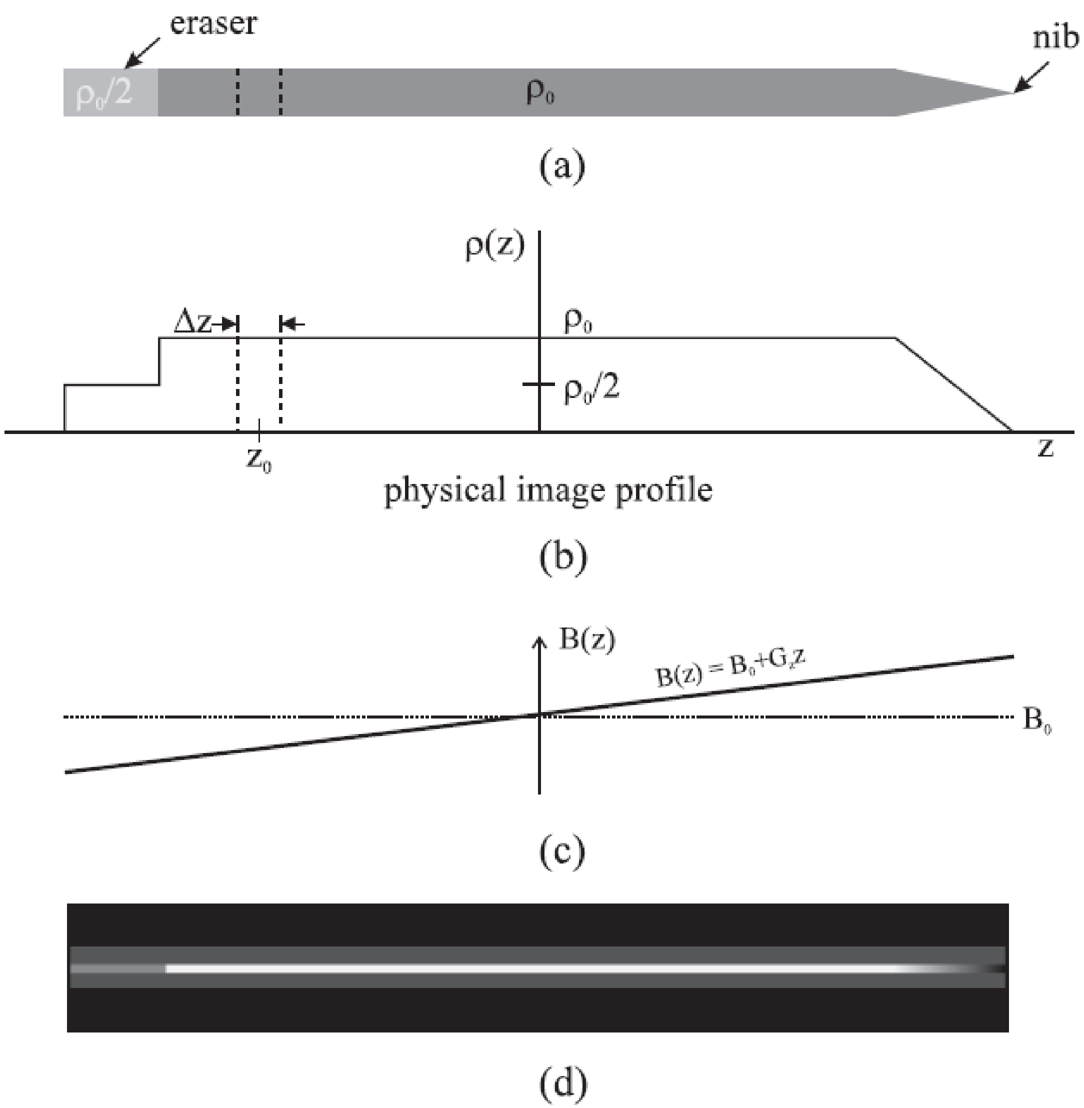
$$\rho(z) = \int_{-\infty}^{\infty} dk 2S_0\cos(2\pi k z_0) e^{i2\pi k z}$$
$$= S_0\int_{-\infty}^{\infty} dk\,\left(e^{i2\pi k(z+z_0)} + e^{i2\pi k(z-z_0)}\right)$$
$$= S_0[\delta(z+z_0) + \delta(z-z_0)]$$



(Fig.9.1)

Pencil model

$$\rho(z) = S_0(z)[\sum_i \delta(z - zi)]$$



(Fig.9.7)

Gradient Echo

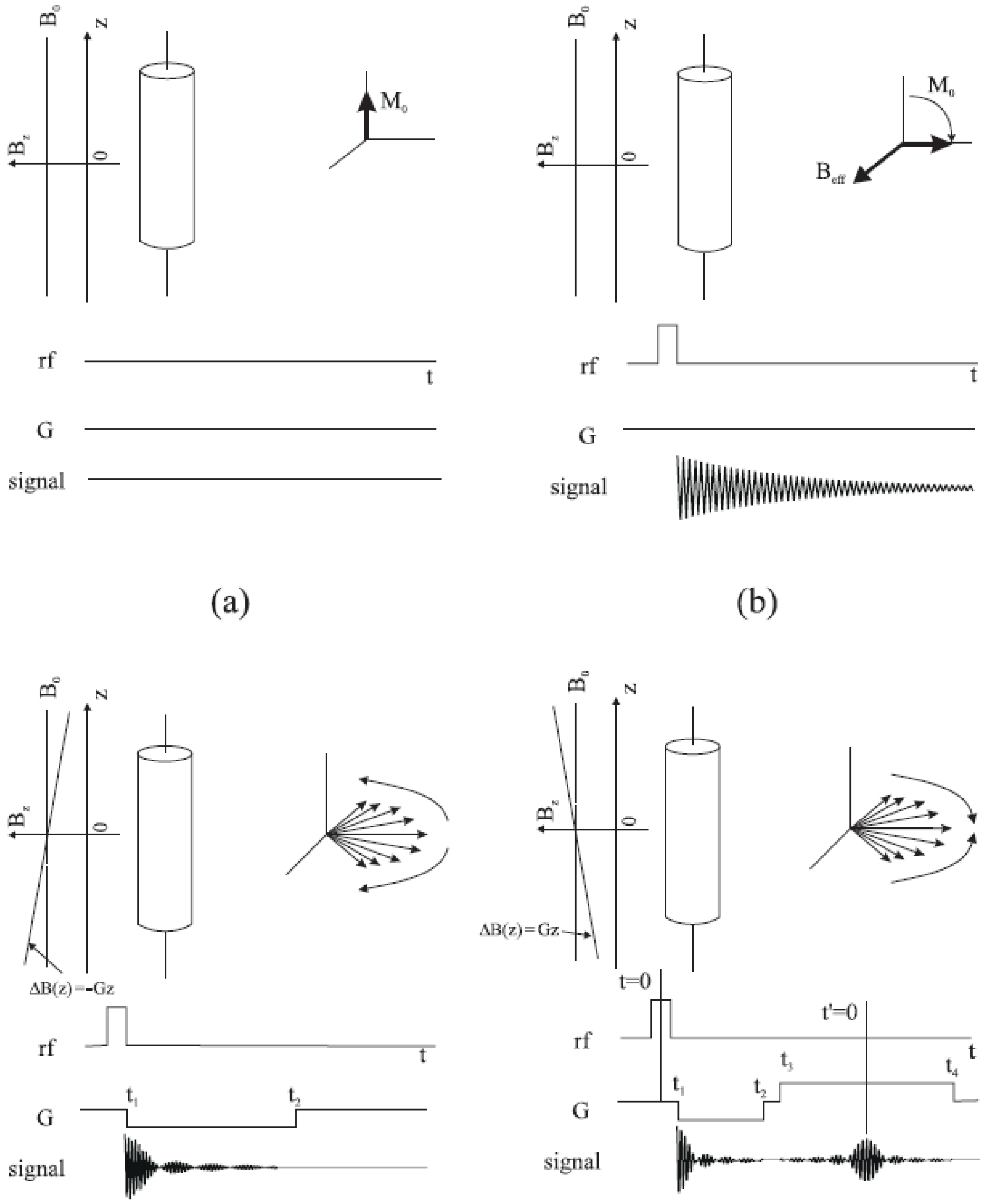
Excitation pulse

Pre-phase gradient

Read gradient

Rule of thumb:

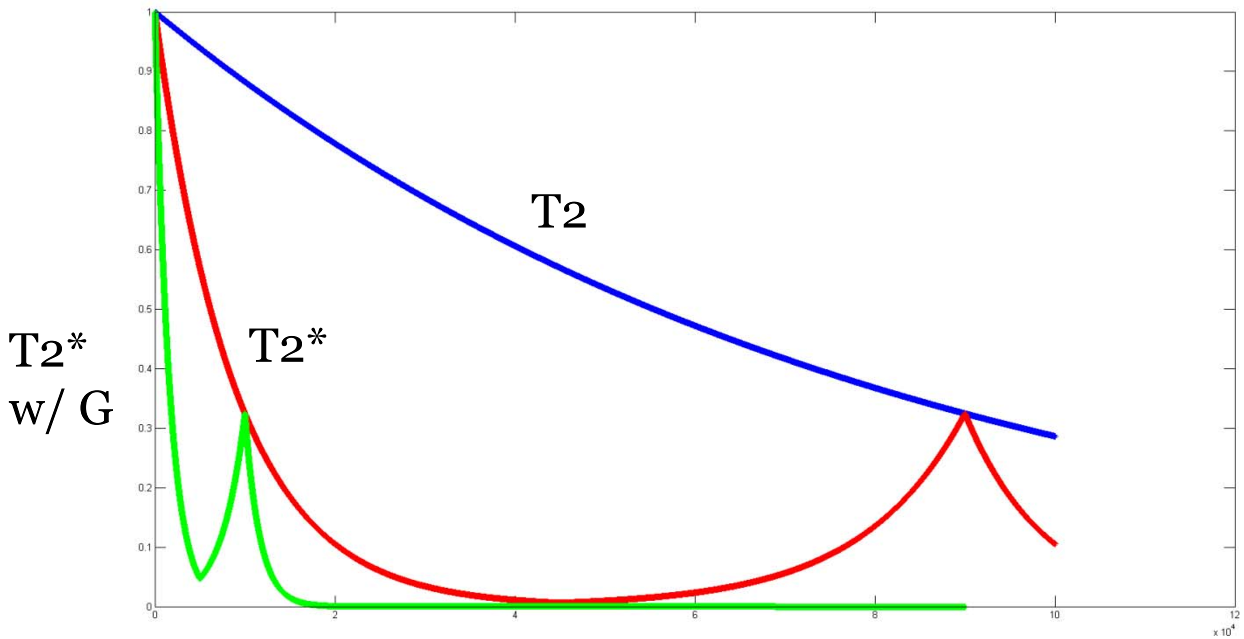
- Net Read gradient moment (i.e. $\int Gt$) is zero at the echo center



(Fig.9.2)

So why Gradient Echo?

- Recover signal from fast T2* decay due to gradients
- Reduce filtering effects (Chap. 13)



So why Gradient Echo?

- Full k-space coverage (Cartesian)

FID
(Half Fourier)

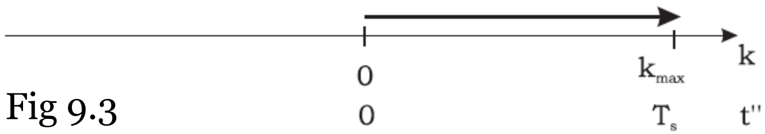
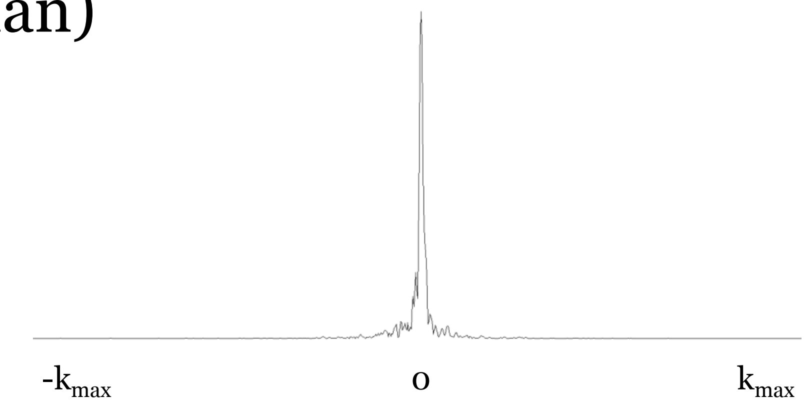


Fig 9.3



GE

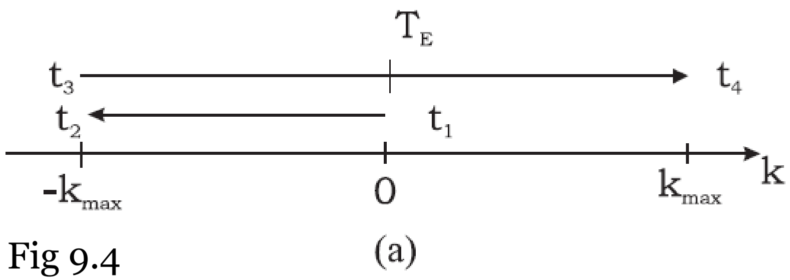


Fig 9.4

$$k(t) \square \gamma \int_0^t dt' G(t'))$$

$$-k_{max} < k < k_{max}$$

Spin Echo 1D k-space

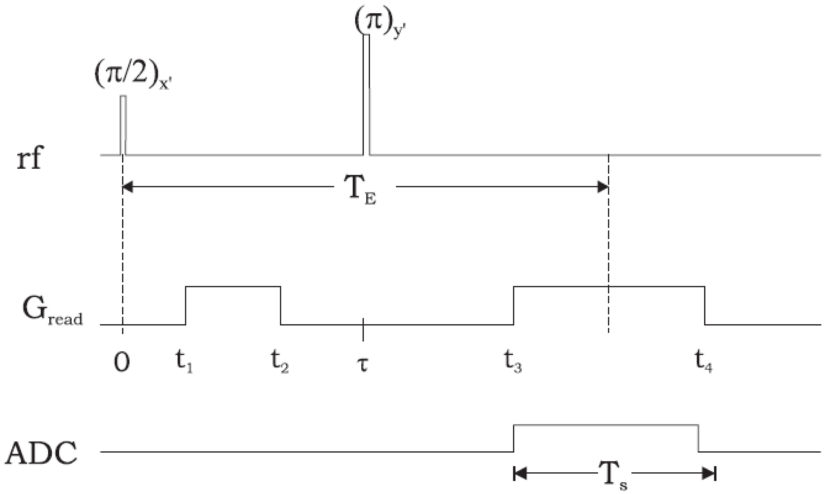
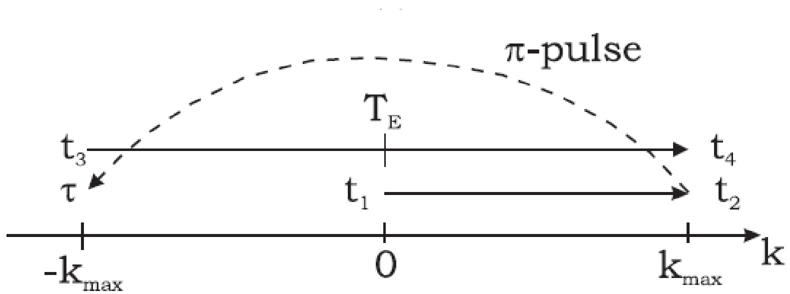


Fig 9.5



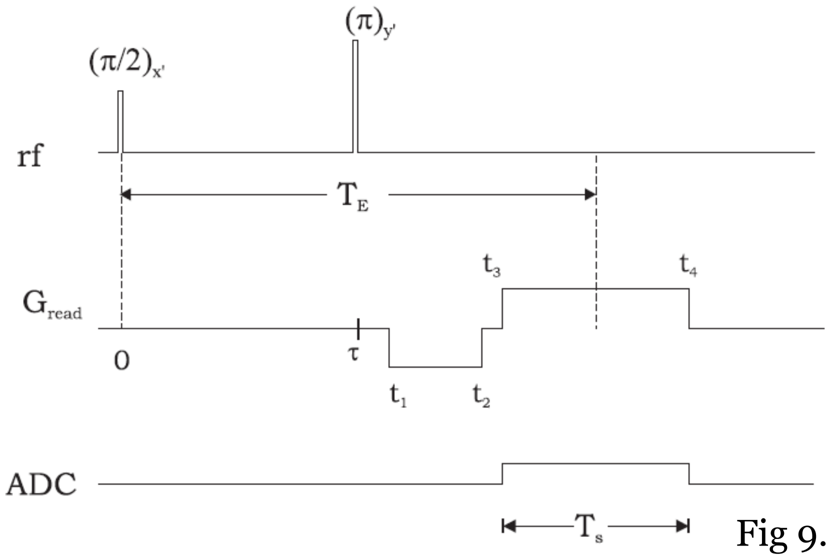


Fig 9.6

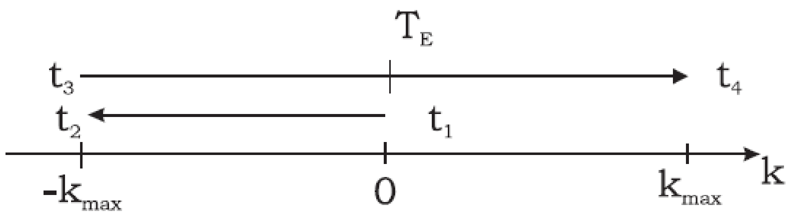


Fig 9.4

Homework

- Prob. *9.1*, *9.2*, *9.4*, *9.5*

Next Class

Chapter 10.1-10.4